Global Exponential Stability of the Periodic Solution of a Discrete-Time Complex-Valued Hopfield Neural Network with Delays and Impulses

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***Abstract*—The global stability of the periodic solution of a discrete-time complex-valued Hopfield neural network is studied. By introducing an appropriate Lyapunov functional it is proved that any two solutions of the system exponentially approach each other with time.**

Keywords-complex neural networks; periodic solution; stability.

I. INTRODUCTION

Over the past three decades, neural networks have been widely studied since they have been successfully applied to various processing problems such as optimization, image processing, associative memory and many other fields (see [10][12] and references given therein). Different types of applications depend on the dynamical behaviors of the neural networks. The existence and stability of equilibrium points and periodic solutions are of particular interest.

In order to solve problems in the fields of optimization, neural control and signal processing, neural networks have to be designed such that there is only one equilibrium point and this equilibrium point is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima. In the case of global stability, there is no need to be specific about the initial conditions for the neural circuits since all trajectories starting from anywhere settle down at the same unique equilibrium. If the equilibrium is exponentially asymptotically stable, the convergence is fast for real-time computations. The unique equilibrium depends on the external stimulus. When the parameters of the neural network and the external stimulus are not constants but periodic functions of time, which is the case in many real-life problems, the role of the equilibrium point is played by a periodic solution.

Numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behavior if either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems [18].

A. Hirose wrote in the introduction to [13]: "Complex-valued neural networks (CVNNs) are effective and powerful in particular to deal with wave phenomena such as electromagnetic and sonic waves, as well as to process wave-related information . . . Researchers extend the world of computation to pattern processing fields based on a novel use of the structure of complex-amplitude (phase and amplitude) information." Further on, he listed the following major application fields of CVNNs: antenna design, estimation of direction of arrival and beamforming of electromagnetic waves, radar imaging, acoustic signal processing and ultrasonic imaging, communications signal processing, image processing, traffic-lights and electric-power systems, quantum devices such as superconductive devices, optical/lightwave information processing including carrier-frequency multiplexing. CVNNs also find applications in fields such as speech synthesis, spatiotemporal analysis of physiological neural devices and systems and artificial neural information processing [23]. CVNNs can be considered as an extension of real-valued neural networks; however, they can be used to solve problems which cannot be solved using their real-valued counterparts [20]. The existence, global asymptotic and exponential stability of equilibrium points of CVNNs have been actively studied in the recent years [6][14][22]. On the other hand, there are very few results on the existence, global asymptotic and exponential stability of periodic solutions of CVNNs [11][21]. These papers deal with delayed CVNNs, respectively of neutral type and with impulses. In [23], sufficient conditions are obtained for the existence and global asymptotic stability of periodic solutions for delayed complex-valued simplified Cohen-Grossberg neural networks.

In our previous paper [7], we constructed a discrete-time counterpart of a complex-valued Hopfield network with time-varying delays and impulses by using the semi-discretization method. We found sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained by using the continuation theorem of coincidence degree theory. The goal of the present paper is to find sufficient conditions for the uniqueness and global exponential stability of the periodic solution of the aforementioned discrete-time system. The motivation for our study was the possibility to apply to CVNNs methods previously applied to real-valued neural network. The exposition is self-contained: its understanding does not require reading of [7].

The rest of the paper is organized as follows: Section II recalls the original continuous-time neural network of [7], its discrete-time counterpart and representation as a real-valued discrete-time neural network of double dimension, and the sufficient conditions for the existence of periodic solutions. In Section III, under some additional conditions including time-independence of the delays, we prove the uniqueness and global exponential stability of the periodic solution by introducing an appropriate Lyapunov functional. More precisely, it is shown that any two solutions of the discrete-time system exponentially approach each other. The proof is more difficult than in the case of real-valued neural networks because of the more complicated form of the Lyapunov functional. Finally, Section IV is Discussion, and Section V is Conclusion and Further Work.

II. PRELIMINARIES

In [7], we consider the following impulsive neural network with time-varying delays:

$$\begin{aligned}\dot{z}_i(t) &= -a_i(t)z_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(z_j(t)) \\ &\quad + \sum_{j=1}^{m} c_{ij}(t)g_j(z_j(t-\tau_{ij}(t))) + I_i(t), \\ &\qquad t>0, \quad t \neq t_k, \qquad (1)\\ \Delta z_i(t_k) &= -\alpha_{ik}z_i(t_k) + \sum_{j=1}^{m} \beta_{ijk}\Phi_j(z_j(t_k)) \\ &\quad + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j(z_j(t_k-\tau_{ij}(t_k))) + \zeta_{ik}, \\ &\qquad k \in \{0\}\cup\mathbb{N}, \qquad (2)\\ z_i(s) &= \varphi_i(s), \quad s\in[-\tau,0], \quad i=\overline{1,m}, \quad (3)\end{aligned}$$

where $z_i(t)$ is the complex-valued state of the i-th neuron at time t; $a_i(t)$ is the rate with which the i-th unit resets its potential to the equilibrium state when isolated from the network and external inputs; $f_j(\cdot)$, $g_j(\cdot)$ denote complex activation functions, respectively without and with delay; the functions $b_{ij}(t)$, $c_{ij}(t)$ represent the weights (or strengths) of the synaptic connections between the j-th neuron and the i-th neuron, respectively without and with transmission delay $\tau_{ij}(t)$; $I_i(t)$ denotes the complex-valued external bias on (input signal introduced from outside the network to) the i-th unit at time t; t_k $(k \in \{0\}\cup\mathbb{N})$ are the moments (instants) of impulse effect satisfying $0=t_0<t_1<t_2<\cdots<t_k<\cdots$ and $\lim_{k\to\infty} t_k = \infty$; $\Delta z_i(t_k) := z_i(t_k+0) - z_i(t_k-0) \equiv z_i(t_k+0) - z_i(t_k)$ represents the instantaneous change of the state of the i-th neuron at time t_k; $\Phi_j(\cdot), \Gamma_j(\cdot) : \mathbb{C}\to\mathbb{C}$ are some functions; α_{ik}, β_{ijk}, γ_{ijk}, ζ_{ik} are some complex constants; and $\tau = \max_{i,j=\overline{1,m}} \sup_{t>0} \tau_{ij}(t)$.

We included a real-life example which is a real-valued neural network of the form (1)–(3) (see, for instance, [1] and [16]):

$$\begin{aligned}C_i\dot{u}_i(t) &= -\frac{u_i(t)}{R_i} + \sum_{j=1}^{m} a_{ij}f_j(u_j(t)) \\ &+ \sum_{j=1}^{m} b_{ij}(t)g_j(u_j(t-\tau_{ij}(t))) + I_i, \quad t>0,\ t\neq t_k, \\ \Delta u_i(t_k) &= J_{jk}(u_i(t_k)), \qquad k\in\mathbb{N}, \\ u_i(s) &= \varphi_i(s), \quad s\in[-\tau,0], \quad i=\overline{1,m},\end{aligned}$$

where $u_i(t)$ denotes the state (voltage) of the i-th neuron at time t, the positive constants C_i and R_i are the neuron amplifier input capacitance and resistance, respectively.

For system (1)–(3) we made the following assumptions:

[H1] There exists a positive number ω and a positive integer p such that

$$a_i(t+\omega) = a_i(t),\ I_i(t+\omega) = I_i(t) \text{ for } t\geq 0 \text{ and } i=\overline{1,m},$$

$$b_{ij}(t+\omega) = b_{ij}(t),\ c_{ij}(t+\omega) = c_{ij}(t),\ \tau_{ij}(t+\omega) = \tau_{ij}(t) \text{ for } t\geq 0 \text{ and } i,j=\overline{1,m},$$

$$t_{k+p} = t_k + \omega \text{ for } k\in\{0\}\cup\mathbb{N},$$

$$\alpha_{i,k+p} = \alpha_{ik},\ \zeta_{i,k+p} = \zeta_{ik} \text{ for } k\in\{0\}\cup\mathbb{N} \text{ and } i=\overline{1,m},$$

$$\beta_{ij,k+p} = \beta_{ijk},\ \gamma_{ij,k+p} = \gamma_{ijk} \text{ for } k\in\{0\}\cup\mathbb{N} \text{ and } i,j=\overline{1,m}.$$

[H2] The complex-valued functions $a_i(t)$, $b_{ij}(t)$, $c_{ij}(t)$ are continuous on $[0,\infty]$; $\operatorname{Re} a_i(t) > 0$ for $t\geq 0$ and $0<\operatorname{Re}\alpha_{ik}<1$ for $k\in\{0\}\cup\mathbb{N}$, $i=\overline{1,m}$.

[H3] There exist positive constants F_j, G_j, $\mathcal{F}_j$, $\mathcal{G}_j$ $(j=\overline{1,m})$ such that

$$\begin{aligned}&\max\{|\operatorname{Re} f_j(u) - \operatorname{Re} f_j(v)|, |\operatorname{Im} f_j(u) - \operatorname{Im} f_j(v)|\} \\ &\leq F_j(|\operatorname{Re} u - \operatorname{Re} v| + |\operatorname{Im} u - \operatorname{Im} v|), \\ &\max\{|\operatorname{Re} g_j(u) - \operatorname{Re} g_j(v)|, |\operatorname{Im} g_j(u) - \operatorname{Im} g_j(v)|\} \\ &\leq G_j(|\operatorname{Re} u - \operatorname{Re} v| + |\operatorname{Im} u - \operatorname{Im} v|), \\ &\max\{|\operatorname{Re}\Phi_j(u) - \operatorname{Re}\Phi_j(v)|, |\operatorname{Im}\Phi_j(u) - \operatorname{Im}\Phi_j(v)|\} \\ &\leq \mathcal{F}_j(|\operatorname{Re} u - \operatorname{Re} v| + |\operatorname{Im} u - \operatorname{Im} v|), \\ &\max\{|\operatorname{Re}\Gamma_j(u) - \operatorname{Re}\Gamma_j(v)|, |\operatorname{Im}\Gamma_j(u) - \operatorname{Im}\Gamma_j(v)|\} \\ &\leq \mathcal{G}_j(|\operatorname{Re} u - \operatorname{Re} v| + |\operatorname{Im} u - \operatorname{Im} v|)\end{aligned}$$

for any $u, v\in\mathbb{C}$.

[H4] The functions $\tau_{ij}(t)$ $(i,j=\overline{1,m})$ are nonnegative and continuous for $t\geq 0$.

[H5] The functions $\varphi_i(s)$ $(i=\overline{1,m})$ are piecewise continuously differentiable on the interval $[-\tau,0]$, with points of possible discontinuity of the form $t_k-\omega$.

To find an ω-periodic solution of system (1), (2) means to determine the initial functions $\varphi_i(s)$ so that the solution of the initial-value problem (1)–(3) is ω-periodic.

In their paper [15] T. Insperger and G. Stépán presented an efficient numerical method for the stability analysis of linear delayed systems. The semi-discretization method is based on discretization with respect to the past effect only. It was shown that the semi-discretization method is much more effective than the full discretization for the stability analysis. The semi-discretization does not preserve the solutions of the original system. However, it does preserve their exponential stability if the semi-discretization is fine enough in some sense.

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A modification of the semi-discretization method was used for the stability analysis of neural networks by S. Mohamad and K. Gopalsamy in [19] and numerous subsequent papers of the same authors. In particular, it can be applied to not necessarily linear neural networks if the nonlinearities satisfy certain conditions.

Similarly to our previous papers [2][3][4], next we derived a discrete counterpart of system (1)–(3) using a modification of the semi-discretization method and obtained sufficient conditions for the existence of periodic solutions of the latter.

For the sake of definiteness we assumed that $\tau \le \omega$. For a positive integer N we chose the discretization step $h = \omega/N$. For the moment we assume N so large that $h < \min\limits_{k=\overline{1,p}}(t_{k+1} - t_k)$. Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denoted $n = [t/h]$, the greatest integer in t/h, $n_k = [t_k/h]$, and $N_0 = [\tau/h]$.

Omitting the details, we present the derived discrete-time counterpart of system (1)–(3):

$$\Delta z_i(n) = -A_i(n)z_i(n) + I_i(n)$$
$$+\begin{cases} \sum\limits_{j=1}^{m} b_{ij}(n)f_j(z_j(n)) + \sum\limits_{j=1}^{m} c_{ij}(n)g_j(z_j(n-\tau_{ij}(n))), \\ \qquad n \ne n_k, \\ \sum\limits_{j=1}^{m} \beta_{ijk}\Phi_j(z_j(n_k)) + \sum\limits_{j=1}^{m} \gamma_{ijk}\Gamma_j(z_j(n_k-\tau_{ij}(n_k))), \\ \qquad n = n_k, \end{cases} \tag{4}$$
$$n \in \{0\} \cup \mathbb{N},$$

$$z_i(s) = \varphi_i(s) \quad \text{for} \quad s = 0, -1, \ldots, -N_0, \quad i = \overline{1,m}, \tag{5}$$

where $z_i(n)$ is the complex-valued state of the i-th neuron at time nh ($n \in \mathbb{Z}$, $n \ge -N_0$; $A_i(n)$ is a complex-valued function with a positive real part; n_k ($k \in \{0\} \cup \mathbb{N}$) are integers satisfying $0 = n_0 < n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim\limits_{k\to\infty} n_k = \infty$; $\Delta z_i(n) := z_i(n+1) - z_i(n)$; $\Phi_j(\cdot), \Gamma_j(\cdot) : \mathbb{C} \to \mathbb{C}$ are some functions; α_{ik}, β_{ijk}, γ_{ijk}, ζ_{ik} are some complex constants; $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s))^T$, $s = 0, -1, \ldots, -N_0$, are given initial vectors, and $N_0 = \max\limits_{i,j=\overline{1,m}} \sup\limits_{n\ge 0} \tau_{ij}(n)$.

From the assumptions H1, H2, H4, it follows that

[H6] There exist positive integers N and p such that

$$A_i(n+N) = A_i(n),\ I_i(n+N) = I_i(n) \ \text{ for } \ i = \overline{1,m}, \quad n \in \{0\} \cup \mathbb{N},$$

$$\tau_{ij}(n+N) = \tau_{ij}(n) \text{ for } i,j = \overline{1,m},\ n \in \{0\} \cup \mathbb{N},$$

$$b_{ij}(n+N) = b_{ij}(n),\ c_{ij}(n+N) = c_{ij}(n) \ \text{ for } \ i,j = \overline{1,m}, \quad n \in \mathbb{N} \setminus \{n_k\}_{k\in\mathbb{N}},$$

$$n_{k+p} = n_k + N \ \text{ for } \ k \in \{0\} \cup \mathbb{N},$$

$$\beta_{ij,k+p} = \beta_{ijk},\ \gamma_{ij,k+p} = \gamma_{ijk} \ \text{ for } \ k \in \{0\} \cup \mathbb{N} \quad \text{and} \quad i,j = \overline{1,m}.$$

[H7] $0 < \operatorname{Re} A_i(n) < 1$ for $i = \overline{1,m}$, $n \in I_N := \{0, 1, \ldots, N-1\}$.

To find an N-periodic solution of system (4) means to determine the initial vectors $\varphi_i(s)$ so that the solution of the initial-value problem (4), (5) is N-periodic. For the sake of definiteness, we assume that $N_0 \le N$.

In order to formulate the main result of [7], we introduced the following notation:

For an N-periodic sequence $v(n)$, we denote $\tilde{v} = \sum\limits_{n=0}^{N-1} v(n)$ (if $v(n)$ is given by a long formula, we write $v\tilde{}$ or $(v)\tilde{}$ instead);

$$\overline{b}_{ij} = \max\{\sup_{n\ne n_k} |\operatorname{Re} b_{ij}(n)|,\ \sup_{n\ne n_k} |\operatorname{Im} b_{ij}(n)|\},$$
$$\overline{c}_{ij} = \max\{\sup_{n\ne n_k} |\operatorname{Re} c_{ij}(n)|,\ \sup_{n\ne n_k} |\operatorname{Im} c_{ij}(n)|\},$$
$$\overline{\beta}_{ij} = \max\{\max_{k=\overline{1,p}} |\operatorname{Re} \beta_{ijk}|,\ \max_{k=\overline{1,p}} |\operatorname{Im} \beta_{ijk}|\},$$
$$\overline{\gamma}_{ij} = \max\{\max_{k=\overline{1,p}} |\operatorname{Re} \gamma_{ijk}|,\ \max_{k=\overline{1,p}} |\operatorname{Im} \gamma_{ijk}|\}, \quad i,j = \overline{1,m};$$

$$\begin{aligned} \rho_i = (N-p)\sum_{j=1}^{m} \big[&\overline{b}_{ij}(|\operatorname{Re} f_j(0)| + |\operatorname{Im} f_j(0)|) \\ &+ \overline{c}_{ij}(|\operatorname{Re} g_j(0)| + |\operatorname{Im} g_j(0)|)\big] \\ +p \sum_{j=1}^{m} \big[&\overline{\beta}_{ij}(|\operatorname{Re} \Phi_j(0)| + |\operatorname{Im} \Phi_j(0)|) \\ &+ \overline{\gamma}_{ij}(|\operatorname{Re} \Gamma_j(0)| + |\operatorname{Im} \Gamma_j(0)|)\big], \quad i = \overline{1,m}; \end{aligned}$$

$$\mathcal{B}_{ij} = 2[(N-p)(\overline{b}_{ij}F_j + \overline{c}_{ij}G_j) + p(\overline{\beta}_{ij}\mathcal{F}_j + \overline{\gamma}_{ij}\mathcal{G}_j)], \quad i,j = \overline{1,m}.$$

Next, we introduced the condition

[H8] $$\min_{i=\overline{1,m}} \left(\widetilde{\operatorname{Re} A_i} - |\operatorname{Im} A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) > 0.$$

We introduce the $m \times m$ matrices

$$\tilde{\mathcal{A}}_R = \operatorname{diag}\left(\widetilde{\operatorname{Re} A_i} \frac{1 - \widetilde{\operatorname{Re} A_i}}{1 + \widetilde{\operatorname{Re} A_i}},\ i = \overline{1,m} \right),$$
$$\tilde{\mathcal{A}}_I = \operatorname{diag}\left(|\operatorname{Im} A_i|\tilde{},\ i = \overline{1,m} \right), \quad \mathcal{B} = (\mathcal{B}_{ij})_{i,j=1}^{m},$$

and the condition

[H9] The $2m \times 2m$ matrix $\mathcal{A} = \begin{pmatrix} \tilde{\mathcal{A}}_R - \mathcal{B} & -\tilde{\mathcal{A}}_I - \mathcal{B} \\ -\tilde{\mathcal{A}}_I - \mathcal{B} & \tilde{\mathcal{A}}_R - \mathcal{B} \end{pmatrix}$ is an M-matrix.

This condition implies that the matrix $\mathcal{A}$ is nonsingular and its inverse has only nonnegative entries [5][8].

The main result of [7] is the following theorem.

Theorem 1: Suppose that conditions H3, H6–H9 hold. Then the system (4) has at least one N-periodic solution.

The theorem was proved using Mawhin's continuation theorem [9, p. 40]. To this end, we denoted $x_i = \operatorname{Re} z_i$, $y_i = \operatorname{Im} z_i$ ($i = \overline{1,m}$), $x = (x_1, x_2, \ldots, x_m)^T$, $y = (y_1, y_2, \ldots, y_m)^T$, and considered $z = (x, y)^T$ as a vector in $\mathbb{R}^{2m}$. Next, we rewrote the complex system (4) as the real system

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$$\Delta x_i(n) = -\mathrm{Re}\, A_i(n)x_i(n) + \mathrm{Im}\, A_i(n)y_i(n) + \mathrm{Re}\, I_i(n)$$
$$+\begin{cases} \sum\limits_{j=1}^{m} [\mathrm{Re}\, b_{ij}(n)\mathrm{Re}\, f_j(z_j(n)) - \mathrm{Im}\, b_{ij}(n)\mathrm{Im}\, f_j(z_j(n)) \\ \quad + \mathrm{Re}\, c_{ij}(n)\mathrm{Re}\, g_j(z_j(n-\tau_{ij}(n))) \\ \quad - \mathrm{Im}\, c_{ij}(n)\mathrm{Im}\, g_j(z_j(n-\tau_{ij}(n)))], \quad n \neq n_k; \\ \sum\limits_{j=1}^{m} [\mathrm{Re}\, \beta_{ijk}\mathrm{Re}\, \Phi_j(z_j(n_k)) - \mathrm{Im}\, \beta_{ijk}\mathrm{Im}\, \Phi_j(z_j(n_k)) \\ \quad + \mathrm{Re}\, \gamma_{ijk}\mathrm{Re}\, \Gamma_j(z_j(n_k-\tau_{ij}(n_k))) \\ \quad - \mathrm{Im}\, \gamma_{ijk}\mathrm{Im}\, \Gamma_j(z_j(n_k-\tau_{ij}(n_k)))], \quad n = n_k, \end{cases} \tag{6}$$

$$\Delta y_i(n) = -\mathrm{Re}\, A_i(n)y_i(n) - \mathrm{Im}\, A_i(n)x_i(n) + \mathrm{Im}\, I_i(n)$$
$$+\begin{cases} \sum\limits_{j=1}^{m} [\mathrm{Re}\, b_{ij}(n)\mathrm{Im}\, f_j(z_j(n)) + \mathrm{Im}\, b_{ij}(n)\mathrm{Re}\, f_j(z_j(n)) \\ + \mathrm{Re}\, c_{ij}(n)\mathrm{Im}\, g_j(z_j(n-\tau_{i-m,j}(n))) \\ + \mathrm{Im}\, c_{ij}(n)\mathrm{Re}\, g_j(z_j(n-\tau_{i-m,j}(n)))], \ n \neq n_k; \\ \sum\limits_{j=1}^{m} [\mathrm{Re}\, \beta_{ijk}\mathrm{Im}\, \Phi_j(z_j(n_k)) + \mathrm{Im}\, \beta_{ijk}\mathrm{Re}\, \Phi_j(z_j(n_k)) \\ + \mathrm{Re}\, \gamma_{ijk}\mathrm{Im}\, \Gamma_j(z_j(n_k-\tau_{i-m,j}(n_k))) \\ + \mathrm{Im}\, \gamma_{ijk}\mathrm{Re}\, \Gamma_j(z_j(n_k-\tau_{i-m,j}(n_k)))], \ n = n_k, \end{cases} \tag{7}$$

for $i = \overline{1, m}$.

In the next section, under some additional assumptions, we prove the global exponential stability of any N-periodic solution of system (4).

III. Main Result

Let us denote

$$B_{ij} = 2\max(\overline{b}_{ij}F_j, \overline{\beta}_{ij}\mathcal{F}_j),$$
$$C_{ij} = 2\max(\overline{c}_{ij}G_j, \overline{\gamma}_{ij}\mathcal{G}_j). \tag{8}$$

Next, we introduce the conditions

[H10] The delays τ_{ij} $(0 \le \tau_{ij} \le N_0)$ are independent of n.

[H11] The inequalities

$$\mathrm{Re}\, A_i(n) - |\mathrm{Im}\, A_i(n)| - \sum_{j=1}^{m}(B_{ji} + C_{ji}) > 0$$

are satisfied for all $n \in I_N$ and $i = \overline{1, m}$.

It is easy to see that condition H11 implies H8.

Our main result is the following theorem.

Theorem 2: Let conditions H3, H6, H7, H10, H11 hold. Let $z^*(n) = (x^*(n), y^*(n))^T$ be an N-periodic solution of system (4). Then there exist constants $M > 1$ and $\overline{\lambda} > 1$ such that for any $\lambda \in (1, \overline{\lambda}]$ and for any other solution $z(n) = (x(n), y(n))^T$ of system (4) defined at least for $n \ge -N_0$ the following estimate holds

$$\sum_{i=1}^{m}(|x_i(n) - x_i^*(n)| + |y_i(n) - y_i^*(n)|) \tag{9}$$
$$\le M\lambda^{-n}\sum_{i=1}^{m}\max_{-N_0 \le s \le 0}(|x_i(s) - x_i^*(s)| + |y_i(s) - y_i^*(s)|)$$

for all $n \in \{0\} \cup \mathbb{N}$.

In the proof of the theorem, we use the following lemma.

Lemma 1: Let condition H11 hold. Then there exists a constant $\overline{\lambda} > 1$ such that for any $\lambda \in (1, \overline{\lambda}]$

$$\lambda\left(1 - \mathrm{Re}\, A_i(n) + |\mathrm{Im}\, A_i(n)| + \sum_{j=1}^{m} B_{ji}\right)$$
$$+ \sum_{j=1}^{m} C_{ji}\lambda^{1+\tau_{ji}} - 1 \le 0$$

for all $n \in I_N$ and $i = \overline{1, m}$.

Proof: Consider the functions

$$\chi_i(n, \lambda) := \lambda\left(1 - \mathrm{Re}\, A_i(n) + |\mathrm{Im}\, A_i(n)| + \sum_{j=1}^{m} B_{ji}\right)$$
$$+ \sum_{j=1}^{m} C_{ji}\lambda^{1+\tau_{ji}} - 1, \quad n \in I_N, \ i = \overline{1, m}.$$

For each $n \in I_N$ and $i = \overline{1, m}$, $\chi_i(n, \lambda)$ is a continuous function of $\lambda \in [1, \infty)$ such that

$$\chi_i(n, 1) = -\left(\mathrm{Re}\, A_i(n) - |\mathrm{Im}\, A_i(n)| - \sum_{j=1}^{m}(B_{ji} + C_{ji})\right) < 0$$

by virtue of condition H11, and $\lim\limits_{\lambda \to \infty} \chi_i(n, \lambda) = +\infty$. Then there exists $\overline{\lambda}_{in} > 1$ such that $\chi_i(n, \overline{\lambda}_{in}) = 0$ and $\chi_i(n, \lambda) \le 0$ for $\lambda \in (0, \overline{\lambda}_{i,n}]$. It suffices to choose $\overline{\lambda} = \max\{\overline{\lambda}_{in} | \ i = \overline{1, m}, \ n \in I_N\}$. ■

Proof of Theorem 2: Let $z^*(n)$ and $z(n)$ be as in the statement of Theorem 2. Our goal will be to construct a Lyapunov functional $V(n)$ of the difference $z(n) - z^*(n)$, which is decreasing with respect to $n \in \{0\} \cup \mathbb{N}$. First, we denote

$$X(n) := x(n) - x^*(n), \qquad Y(n) := y(n) - y^*(n).$$

Then, from (6) for $n \in \{0\} \cup \mathbb{N}$, $n \neq n_k$, we have

$$X_i(n+1) = (1 - \mathrm{Re}\, A_i(n))X_i(n) + \mathrm{Im}\, A_i(n)Y_i(n)$$
$$+ \sum_{j=1}^{m}\{\mathrm{Re}\, b_{ij}(n)[\mathrm{Re}\, f_j(z_j(n)) - \mathrm{Re}\, f_j(z_j^*(n))]$$
$$- \mathrm{Im}\, b_{ij}(n)[\mathrm{Im}\, f_j(z_j(n)) - \mathrm{Im}\, f_j(z_j^*(n))]\}$$
$$+ \sum_{j=1}^{m}\{\mathrm{Re}\, c_{ij}(n)[\mathrm{Re}\, g_j(z_j(n-\tau_{ij})) - \mathrm{Re}\, g_j(z_j^*(n-\tau_{ij}))]$$
$$- \mathrm{Im}\, c_{ij}(n)[\mathrm{Im}\, g_j(z_j(n-\tau_{ij})) - \mathrm{Im}\, g_j(z_j^*(n-\tau_{ij}))]\}$$

and, by virtue of H3, we derive

$$|X_i(n+1)|$$
$$\le (1 - \mathrm{Re}\, A_i(n))|X_i(n)| + |\mathrm{Im}\, A_i(n)|\,|Y_i(n)|$$
$$+ \sum_{j=1}^{m} 2\overline{b}_{ij}F_j(|X_j(n)| + |Y_j(n)|)$$
$$+ \sum_{j=1}^{m} 2\overline{c}_{ij}G_j(|X_j(n-\tau_{ij})| + |Y_j(n-\tau_{ij})|). \tag{10}$$

In a similar way, we obtain

$$\begin{aligned} &|X_i(n_k+1)| \\ &\leq (1-\operatorname{Re} A_i(n_k))|X_i(n_k)| + |\operatorname{Im} A_i(n_k)|\,|Y_i(n_k)| \\ &+\sum_{j=1}^{m} 2\overline{\beta}_{ij}\mathcal{F}_j(|X_j(n_k)|+|Y_j(n_k)|) \\ &+\sum_{j=1}^{m} 2\overline{\gamma}_{ij}\mathcal{G}_j(|X_j(n_k-\tau_{ij})|+|Y_j(n_k-\tau_{ij})|). \end{aligned} \tag{11}$$

Using the notation (8), inequalities (10) and (11) can be written by one formula as

$$\begin{aligned} &|X_i(n+1)| \\ &\leq (1-\operatorname{Re} A_i(n))|X_i(n)| + |\operatorname{Im} A_i(n)|\,|Y_i(n)| \\ &+\sum_{j=1}^{m} B_{ij}(|X_j(n)|+|Y_j(n)|) \\ &+\sum_{j=1}^{m} C_{ij}(|X_j(n-\tau_{ij})|+|Y_j(n-\tau_{ij})|) \end{aligned} \tag{12}$$

for all $n \in \{0\}\cup\mathbb{N}$.

Similarly, from (7) we derive

$$\begin{aligned} &|Y_i(n+1)| \\ &\leq (1-\operatorname{Re} A_i(n))|Y_i(n)| + |\operatorname{Im} A_i(n)|\,|X_i(n)| \\ &+\sum_{j=1}^{m} B_{ij}(|X_j(n)|+|Y_j(n)|) \\ &+\sum_{j=1}^{m} C_{ij}(|X_j(n-\tau_{ij})|+|Y_j(n-\tau_{ij})|) \end{aligned} \tag{13}$$

for all $n \in \{0\}\cup\mathbb{N}$.

Next, we define the quantities

$$W_i(x) = \lambda^n|X_i(n)|, \qquad \Psi_i(n) = \lambda^n|Y_i(n)|$$

for $\lambda \in (1,\overline{\lambda}]$, $n \geq -N_0$ and $i=\overline{1,m}$. Then, in view of (12) and (13), we obtain

$$\begin{aligned} &W_i(n+1) \\ &\leq \lambda(1-\operatorname{Re} A_i(n))W_i(n) + \lambda|\operatorname{Im} A_i(n)|\Psi_i(n) \\ &+\lambda\sum_{j=1}^{m} B_{ij}(W_j(n)+\Psi_j(n)) \\ &+\sum_{j=1}^{m} C_{ij}\lambda^{1+\tau_{ij}}[W_j(n-\tau_{ij})+\Psi_j(n-\tau_{ij})], \end{aligned} \tag{14}$$

$$\begin{aligned} &\Psi_i(n+1) \\ &\leq \lambda(1-\operatorname{Re} A_i(n))\Psi_i(n) + \lambda|\operatorname{Im} A_i(n)|W_i(n) \\ &+\lambda\sum_{j=1}^{m} B_{ij}(W_j(n)+\Psi_j(n)) \\ &+\sum_{j=1}^{m} C_{ij}\lambda^{1+\tau_{ij}}[W_j(n-\tau_{ij})+\Psi_j(n-\tau_{ij})]. \end{aligned} \tag{15}$$

Inequalities (14), (15) suggest us to define the Lyapunov functional

$$V(n) = \sum_{i=1}^{m}\left[W_j(n)+\Psi_j(n) + \sum_{j=1}^{m} C_{ij}\lambda^{1+\tau_{ij}}\sum_{s=n-\tau_{ij}}^{n-1}(W_j(s)+\Psi_j(s))\right]$$

for all $n \in \{0\}\cup\mathbb{N}$. Then, we have

$$\begin{aligned} V(n+1) &= \sum_{i=1}^{m}\Bigg[W_j(n+1)+\Psi_j(n+1) \\ &+\sum_{j=1}^{m} C_{ij}\lambda^{1+\tau_{ij}}\sum_{s=n+1-\tau_{ij}}^{n}(W_j(s)+\Psi_j(s))\Bigg] \\ &\leq \sum_{i=1}^{m}\Bigg\{\lambda\Bigg[(1-\operatorname{Re} A_i(n)+|\operatorname{Im} A_i(n)|)(W_i(n)+\Psi_i(n)) \\ &\qquad+\sum_{j=1}^{m} B_{ij}(W_j(n)+\Psi_j(n))\Bigg] \\ &+\sum_{j=1}^{m} C_{ij}\lambda^{1+\tau_{ij}}\sum_{s=n-\tau_{ij}}^{n}(W_j(s)+\Psi_j(s))\Bigg\} \end{aligned}$$

and

$$\begin{aligned} \Delta V(n) &\leq \sum_{i=1}^{m}\Bigg\{\lambda\left[1-\operatorname{Re} A_i(n)+|\operatorname{Im} A_i(n)|+\sum_{j=1}^{m} B_{ji}\right] \\ &\qquad+\sum_{j=1}^{m} C_{ji}\lambda^{1+\tau_{ji}}-1\Bigg\}(W_i(n)+\Psi_i(n)) \\ &= \sum_{i=1}^{m}\chi_i(n,\lambda)(W_i(n)+\Psi_i(n)) \leq 0 \end{aligned}$$

in view of Lemma 1. This means that $V(n+1)\leq V(n)$ for all $n\in\{0\}\cup\mathbb{N}$. In particular,

$$V(n)\leq V(0) \quad \text{for all} \quad n\in\{0\}\cup\mathbb{N} \quad \text{and} \quad \lambda\in(1,\overline{\lambda}].$$

Taking into account that

$$V(n) \geq \lambda^n\sum_{i=1}^{m}(|x_i(n)-x_i^*(n)|+|y_i(n)-y_i^*(n)|)$$

and

$$\begin{aligned} V(0) &= \sum_{i=1}^{m}\Bigg[|x_i(0)-x_i^*(0)|+|y_i(0)-y_i^*(0)| \\ &+\sum_{j=1}^{m} C_{ji}\lambda^{1+\tau_{ji}}\sum_{s=-\tau_{ji}}^{-1}(|x_i(s)-x_i^*(s)|+|y_i(s)-y_i^*(s)|)\Bigg] \\ &\leq \max_{i=\overline{1,m}}\left(1+\overline{\lambda}^{1+N_0}\sum_{j=1}^{m} C_{ji}\right) \\ &\times\sum_{i=1}^{m}\max_{-N_0\leq s\leq 0}(|x_i(s)-x_i^*(s)|+|y_i(s)-y_i^*(s)|), \end{aligned}$$

we derive the estimate (9) with

$$M = \max_{i=\overline{1,m}} \left(1 + \overline{\lambda}^{1+N_0} \sum_{j=1}^{m} C_{ji} \right).$$

The proof of this estimate did not use the assumption that the solution $z^*(n)$ is N-periodic. In fact, it shows that system (4) can have at most one N-periodic solution and such a solution is globally exponentially stable. ■

IV. DISCUSSION

Our previous experience with papers devoted to neural networks has shown us that most of these papers can be assigned to one of two quite distinct classes — theoretical and applied (practical).

The papers of the first class usually list some real-life applications in their introductions. These applications are normally taken from surveys on neural networks or the introductions of other papers of the same class. Then, the authors study a mathematical model, which is usually a far-going generalization of an application of neural networks to a real-life problem. The properties of the mathematical model are examined using methods, often much more complicated than in the present paper. Finally, a few examples of low-dimensional neural networks satisfying the conditions obtained may be given, and some computations may be carried out. However, applications of the results obtained to real-life problems are very seldom given.

The papers of the second class are usually devoted to a quite concrete real-life problem, say, the identification of people by their fingerprints. Experimental data are usually given, but very little mathematics is used and models to be studied by papers of the first class are seldom given.

The present paper, as well as our previous papers devoted to neural networks, belong to the first class. So it is not easy to give applications to real-life problems.

To the best of our knowledge, the above mentioned two classes of papers grow (maybe exponentially) quite independently of each other. We hope that a cooperation between "theoreticians" and "practicians" could prove fruitful for both trends.

V. CONCLUSION AND FUTURE WORK

In the present paper, we obtained sufficient conditions for any two solutions of a discrete-time complex-valued Hopfield neural network with delays and impulses to infinitely approach each other with time. The proof was accomplished by constructing an appropriate Lyapunov functional. The result obtained implies the uniqueness and global exponential stability of a periodic solution, provided that it exists.

In future, in the theoretical aspect, we can extend our research to quaternionic neural networks, which are a generalization of CVNNs. On the other hand, in case of an available "practician" as a co-author, we can concentrate on finding real-life examples and applications of the CVNNs considered in the present paper and the results obtained.

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